Definition: Critical Points

A critical point are the points that occur/exist on $f(x)$ and where $f'(x) = 0$ or when $f'(x)$ is undefined .

To find critical points, from an equation, we would first find $f'(x)$ and solve when $f'(x) = 0$ and also when $f'(x)$ is undefined (division by zero, ln of zero, etc…) **and** see if these values are part of the domain of the original function.

We should note that we may need to common factor strange things (like $x^{-1/2}e^x + x^{1/2}e^x$) which we can do by common factoring the lowest exponent (the negative exponent that is the most negative) and using exponent laws to determine the remaining factor(s).

Example 1:

Determine all critical x-values given the function: $f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$

Solution:

We first note that the domain of this function is $(-\infty, \infty)$ as this is a polynomial. We then find the derivative to identify critical points and factor the expression as much as possible.

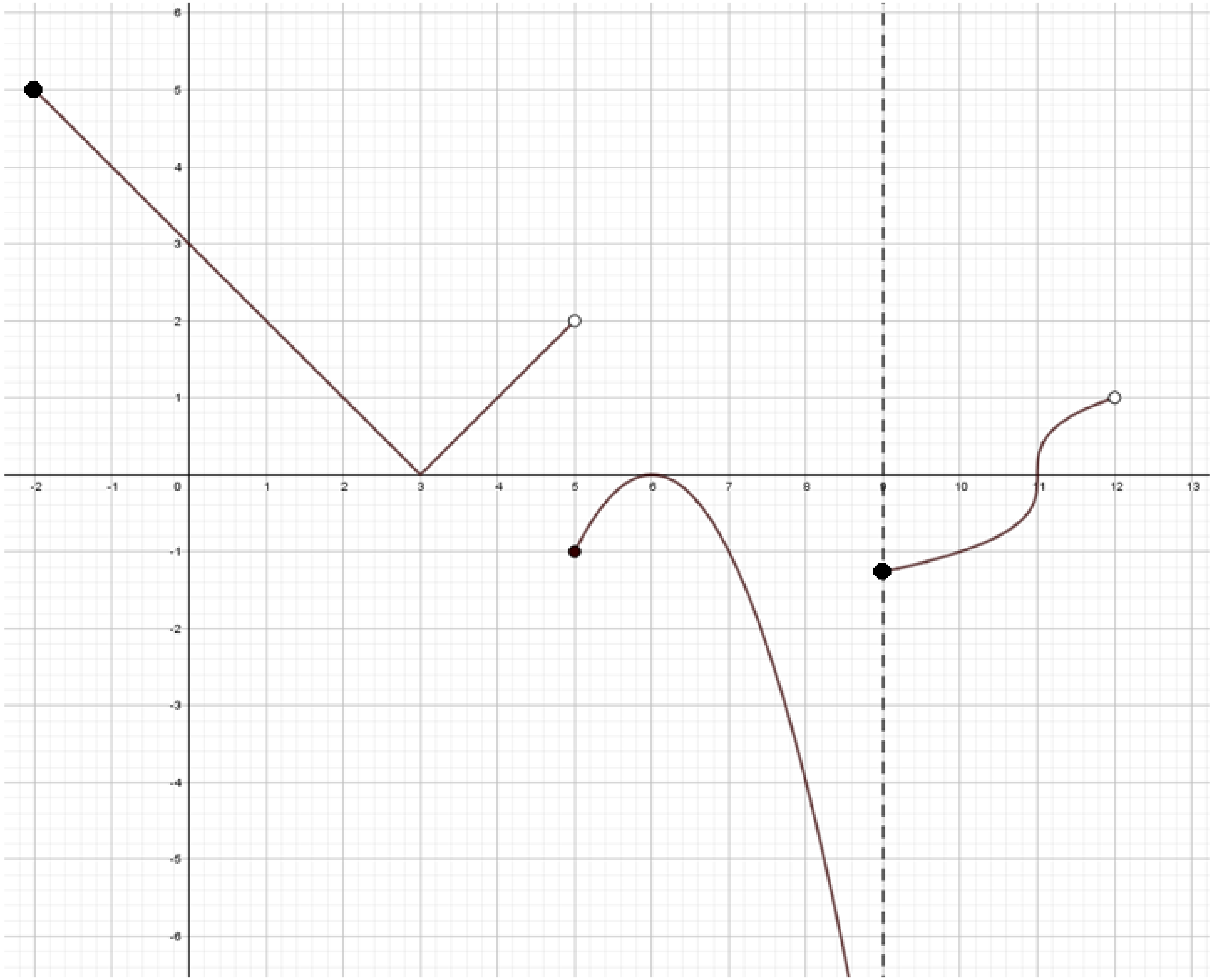
$$\begin{aligned} f'(x) &= x^3 - 6x^2 + 8x \\ &= x(x^2 - 6x + 8) \\ &= x(x-4)(x-2) \end{aligned}$$

Since there is nothing where the derivative is undefined, all of our critical points occur when the derivative is zero. This means this will occur when $x = 0, x = 2, x = 4$.

Examples: Critical Points

Example 2:

Given the graph below, determine the x-values where critical points occur:



Solution:

We look for points where the derivative does not exist (abrupt changes in slope, endpoints, vertical tangents, or points of discontinuity) and where the slope of the tangent is 0. We also need to check to make sure a point exists at that x value (ie there is no hole/discontinuity there).

This happens when:

$x = -2$ (endpoint)

$x = 3$ (abrupt change in slope)

$x = 5$ (jump discontinuity)

$x = 6$ (Slope of tangent is 0)

$x = 9$ (discontinuity)

$x = 11$ (Vertical Tangent)

We note that even though the tangent is undefined at x=12, this is not where a critical point occurs, as there is no point there!

Examples: Critical Points

Example 3:

Determine all critical x-values given the function: $f(x) = \frac{\ln(x)}{x}$

Solution:

We first note that the domain of this function is $(0, \infty)$ as we cannot log 0 or negative numbers, and we cannot divide by 0. We differentiate to find critical points:

$$
\begin{aligned}
f'(x) &= \frac{A'B - AB'}{B^2} \\
&= \frac{\frac{1}{x}(x) - \ln(x)(1)}{x^2} \\
&= \frac{1 - \ln(x)}{x^2}
\end{aligned}
$$

We note that the derivative is undefined when $x = 0$ due to the division by x^2, but the function is also undefined when $x = 0$ so there is no critical point at $x = 0$. We note that the derivative is 0 when the numerator $1 - \ln(x) = 0$ This happens when $\ln(x) = 1$ or (when we e both sides) when $x = e$. This means we have a critical x value at $x = e$.

Examples: Critical Points

Example 4:

Determine all critical x-values given the function: $f(x) = \sqrt[3]{x}e^x$

Solution:

We first note that the domain of this function is $(-\infty, \infty)$ as we can cube root and exponentiate any number. We then find the derivative to identify critical points and factor the expression as much as possible.

$$\begin{aligned} f(x) &= AB \qquad\qquad (A = x^{1/3}\ B = e^x) \\ f'(x) &= A'B + AB' \\ &= \frac{1}{3}x^{-2/3}e^x + x^{1/3}e^x \\ &= x^{-2/3}e^x\left[\frac{1}{3} + x\right] \end{aligned}$$

- Here we see that we have an undefined tangent when $x = 0$ (due to the $x^{-2/3}$ would produce a 0 in the denominator). Thus we have one critical x value when $x = 0$ (it is in the domain of $f(x)$)

- We cannot have e^x = 0 (as the exponential function is always above the x-axis). It is also defined everywhere, so it does not contribute any critical x-values.

- We also have $\frac{1}{3} + x = 0$ when $x = -\frac{1}{3}$, thus we have another critical value at $x = -\frac{1}{3}$.

Thus our critical x-values occur at $x = -\frac{1}{3}$ and at $x = 0$.

Definition: Local and Absolute Extreme Points

A local minimum, is a point that has the smallest y-value from any points close to the minimum.

A local maximum, is a point that has the largest y-value from any points close to the maximum.

An absolute minimum value over a given interval is the smallest y value the function obtains on the entire interval.

An absolute maximum value over a given interval is the largest y value the function obtains on the entire interval.

Examples: Local and Absolute Extreme Points

Example 5:

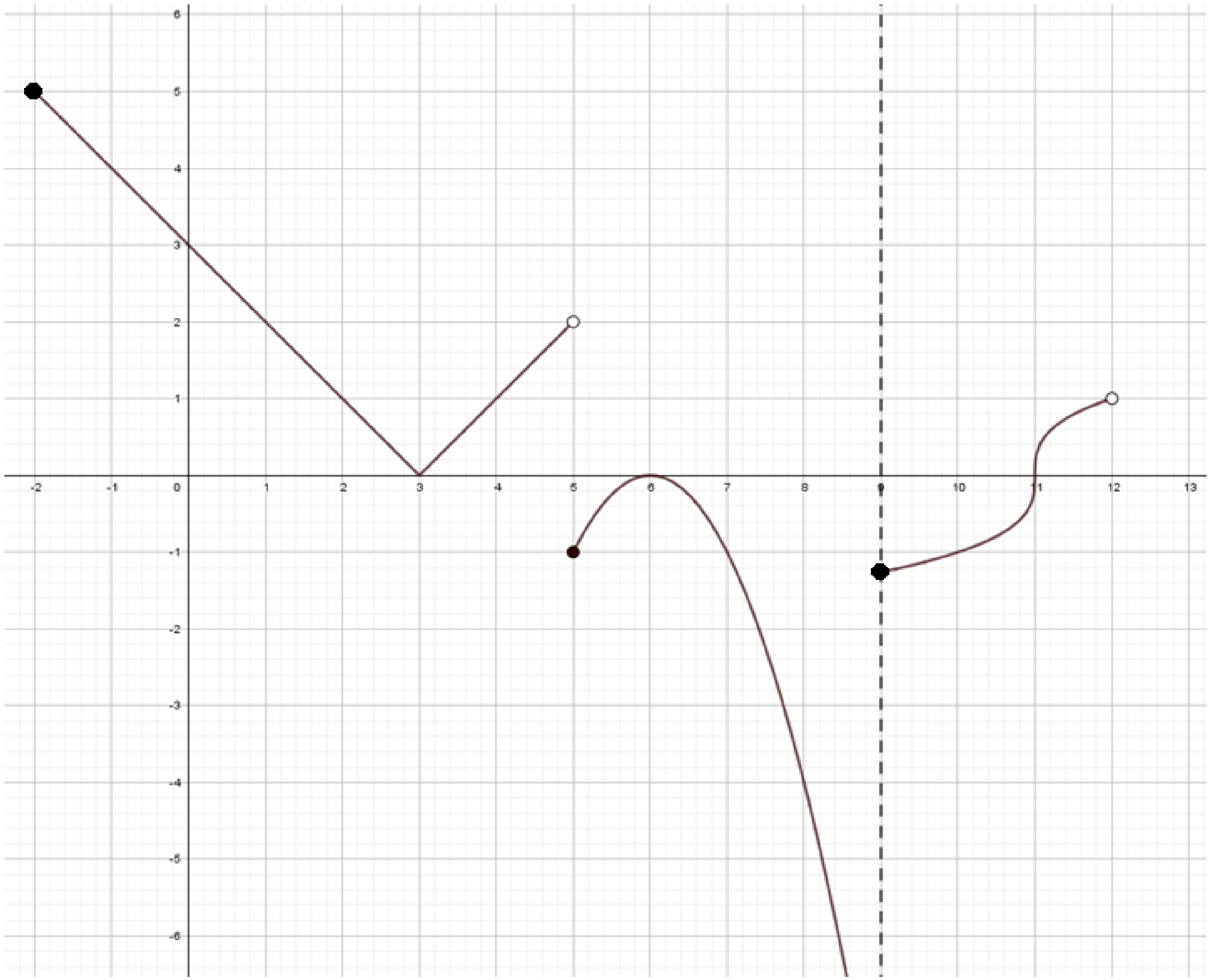
Given the graph below, determine:

a) The local minimum points (if any)

b) The local maximum points (if any)

c) The absolute maximum point (if any)

d) The absolute minimum point (if any)



Solution:

a) Local minimums happen at (3,0) and (5,-1) as points near on the left and right of these local mins all have larger y-values.

b) Local maximums happen at (-2,5) and (6,0) as points near on the left and right of these local mins all have smaller y-values.

c) The absolute maximum happens at (-2,5) as 5 is the largest y-value that appears on the graph.

d) There is no absolute minimum as there is an asymptote that has the graph tending towards negative infinity.

Strategy: Extreme Value Theorem:

Extreme Value Theorem:

Given:

1. $f(x)$ is a <u>continuous</u> function on a closed interval $\underline{[a, b]}$

Conclusion:

$f(x)$ must have an absolute maximum and an absolute minimum on $[a, b]$ and the only candidates are:

$f(a)$, $f(b)$ or at one of the critical points.

<u>**How To Use it:**</u>

Determine $f'(x)$ and find:

1) All critical points when $f'(x) = 0$

2) All critical points when $f'(x)$ is undefined.

3) $f(a)$ and $f(b)$ and $f(critical\ points)$

The largest number in step 3 is the absolute maximum, and the smallest number in step 3 is the absolute minimum in the interval $[a. b]$

<u>When To Use it:</u>

When you want to find absolute maximums or minimums on a closed interval for continuous functions.

<u>Why this works?</u>

See a proof <u>here</u>, but you will need to understand what it means to have a least upper bound (and proof by contradiction).

Examples: Extreme Value Theorem

Example 7:

Find the absolute maximum and minimum on $[-1,2]$ for the function $f(x) = 2^{-x^2}$

Solution:

We first sub in the end points to get: $f(-1) = 2^{-1} = \frac{1}{2}$ and $f(2) = 2^{-4} = \frac{1}{16}$

Next we find critical points by calculating the derivative: $f'(x) = [-2(\ln(2))x]2^{-x^2}$

Here we see that the derivative has no undefined derivatives as we have no division by zeros that can happen (even though $2^{-x^2} = \frac{1}{2^{x^2}}$ we know that $2^{\square}$ is always higher than 0 as exponentials do not have x-intercepts, so we will not have anything undefined here).

We look when each piece can be equal to 0, in this case $-2\ln(2)\,x = 0$ only when $x = 0$ and 2^{-x^2} is never 0, so we only have $x = 0$ that gives a critical point. When $x = 0, f(0) = 2^{-0^2} = 1.$

This means the absolute maximum on the interval is 1 and the absolute minimum on the interval is $\frac{1}{16}$.

Examples: Extreme Value Theorem

Example 8:

Find the absolute maximum and minimum on $[-3,4]$ for the function $f(x) = |x^2 + 4x|$

Solution:

We first sub in the end points to get: $f(-3) = |(-3)^2 + 4(-3)| = 3$ and $f(4) = |4^2 + 4(4)| = 32$

Next we find critical points by calculating the derivative: $f'(x) = \frac{x^2+4x}{|x^2+4x|}(2x + 4)$

Here we see that we have undefined derivatives when $x^2 + 4x = 0$ (due to the division by zero). If we factor this we get $x(x + 4) = 0$. Thus we test x values of $x = 0$ and $x = -4$. Since only one of these are in the interval, we only test $x = 0$.
$f(0) = |0^2 + 4(0)| = 0$

Next, We look when each piece can be equal to 0, we cannot test the first numerator (as they were the values that made the derivative undefined) but instead we note that we would solve $2x + 4 = 0$ which gives us $x = -2$ to test. $f(-2) = |(-2)^2 + 4(-2)| = 4$

This means the absolute maximum on the interval is 32 and the absolute minimum on the interval is 0.

Examples: Extreme Value Theorem

Example 9:

a) Find/draw a function that is not continuous that does not obtain an absolute max nor an absolute min on a closed interval $[-3,3]$

b) Find/draw a function that is not continuous that still does obtains an absolute max and an absolute min on a closed interval $[-3,3]$

Solution:

We have many ways to do this, but one way could be to draw a sin function that has holes at the maximum and minimum values. The function would be discontinuous, but not have an absolute maximum point (or minimum point) as there is no point there. We could get really close to the max value of 1 and min value of -1, but we could never reach them.

We have many ways to do this, but one way could be to draw a sin function that has holes on the x intercepts. This would be discontinuous in the interval, but still reach the max and min values of 1 and -1.

We note that this means having a discontinuous function means "more work required" since it is possible to think of functions that do have absolute extreme points and other functions that do not have absolute extreme points.

Strategy: Finding Intervals of Increasing/Decreasing and Using the First Derivative Test

How To Use it:	When To Use it:	Why this works?
To find the intervals of increase/decrease we can: 1) Identify the domain of the function. 2) Identify all x values for the points of discontinuity. 3) Determine $f'(x)$. 4) Factor the expression fully and find all critical x-values. 5) Create an interval table for $f'(x)$ (not $f(x)$) where we plot the domain, points of discontinuity, and critical points as the column separators, and the factors of the derivative as the rows. The product will tell us if the value is positive or negative which determines if the function is increasing (+) or decreasing (-). 6) If a critical point comes from increasing then goes to decreasing, it is a local maximum. If it comes from decreasing and goes to increasing it is a local minimum. If it is anything else, it is neither a local maximum or minimum. Note: We list the intervals instead of using union as we want to avoid confusion about "always increasing" or "always decreasing" through the union of the interval.	When we want to find when a function is increasing or decreasing and/or classify critical points.	We recall that increasing means that the derivative is positive and decreasing means that the derivative is negative. This strategy just gives us a list of steps to identify all of these places by looking at the derivative. As for classifying local maximums or minimums, we should see that maximums come in on increasing then switch to decreasing. Similarly, minimums come in on decreasing and switch to increasing.

Examples: Classifying Extreme Points

Example 10:

Determine the intervals of increasing and decreasing and all critical points for the following function. Use the first derivative test to determine if the critical points are local maximums or minimums: $f(x) = \frac{x^3}{3x^2+1}$

Solution:

We note that the domain of the function is $(-\infty, \infty)$ as although we have a division, $3x^2 + 1$ is never zero (if you try to solve it, you will get a square root of a negative number). This means there are no points of discontinuity.

Next, we find the derivative:

$$\begin{aligned} f'(x) &= \frac{A'B-AB'}{B^2} \\ &= \frac{3x^2(3x^2+1)-x^3(6x)}{(3x^2+1)^2} \\ &= \frac{3x^4+3x^2}{(3x^2+1)^2} \\ &= \frac{3x^2(x^2+1)}{(3x^2+1)^2} \end{aligned}$$

Here we see that we do not have any undefined derivatives (denominator is never zero) but we do have a critical point when the numerator is zero (this happens when $x = 0$ as $x^2 + 1$ can never be zero). Thus we only have one critical x-value at $x = 0$. Since we want a point, we solve when $y = \frac{0^3}{3(0)^3+1} = 0$. Thus our only critical point is at $(0,0)$.

Finally we can create an interval table to determine the areas of increasing and decreasing and also classify the critical point:

$-\infty$ 0 ∞

f' factors	-1	1
$3x^2$	+	+
$x^2 + 1$	+	+
$3x^2 + 1$	+	+
Product	+	+

Thus the function is always increasing, and the critical point is not a max nor a min.

Examples: Classifying Extreme Points

Example 11:

Consider a function whose derivative is given as $f'(x) = \frac{e^x(x-1)(x+1)(\ln(x)-1)}{\sqrt[3]{(x-5)}^2}$ If we assume that $f(x)$ has a domain of $(0, \infty)$, then:

A) Find all critical x-values
B) Determine the intervals of increasing and decreasing.
C) Determine which critical points would be local maximums, minimums or neither.

Solution:

We have the domain and the derivative, so we simply must find the critical x-values:

- We see that e^x can never be zero, so it contributes no critical values.
- $x = 1$ and $x = -1$ both contribute critical values, but only $x = 1$ is in the domain and thus contributes a critical value.
- $\ln(x) - 1 = 0$ when $\ln x = 1$ which solves to become $x = e$ that is in the domain and contributes a critical value.
- $x = 5$ is also an undefined derivative which is in the domain of f, so it contributes a critical value.
- ln x cannot be 0 so it is an undefined derivative, but is not a critical value as it is not in the domain.

∴ Our critical x-values are $x = 1, e$, and 5.

Examples: Classifying Extreme Points

Example 11 (cont):

Consider a function whose derivative is given as $f'(x) = \frac{e^x(x-1)(x+1)(\ln(x)-1)}{\sqrt[3]{(x-5)}^2}$ If we assume that $f(x)$ has a domain of $(0, \infty)$, then:

Solution:

Using the critical x-values we found, we can now create out interval table:

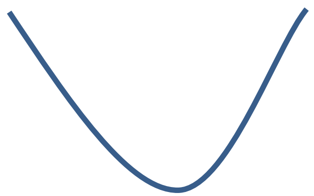
	0 1	1 e	e 5	5 ∞
f' **factors**	**0.5**	**2**	**3**	**6**
e^x	+	+	+	+
$x-1$	-	+	+	+
$x+1$	+	+	+	+
$\ln(x)-1$	-	-	+	+
$\sqrt[3]{(x-5)}^2$	+	+	+	+
product	+	-	+	+

Thus the function is increasing on $(0,1)$ and on (e, ∞) and decreasing on $(1, e)$

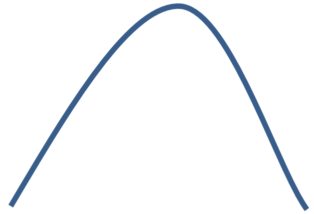
This also means that $x = 1$ will be a local maximum, and $x = e$ will be a local minimum. At $x = 5$ it is neither a local maximum or minimum.

Definition: Concavity and Inflection Points

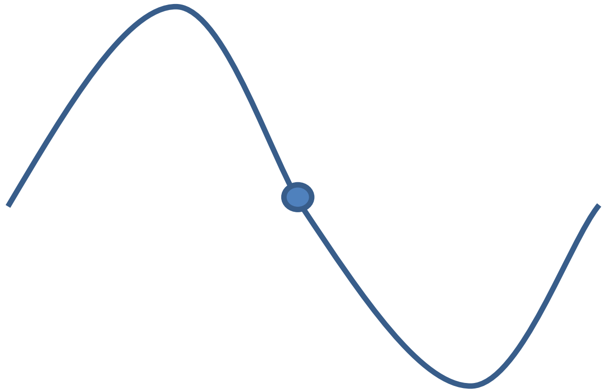
We say that a function is <u>concave up</u> when the point appears to be on a <u>valley</u>. For example, parts of the graph that look like:



We say that a function is <u>concave down</u> when the point appears to be on a <u>hill</u>. For example, parts of the graph that look like:



We call an <u>inflection point</u> a point that changes from concave up to concave down (or vice versa) on a continuous function. (Note this means that endpoints cannot be inflection points).



We note that a straight line does not have any concavity, as it does not curve in any way.

Examples: Concavity

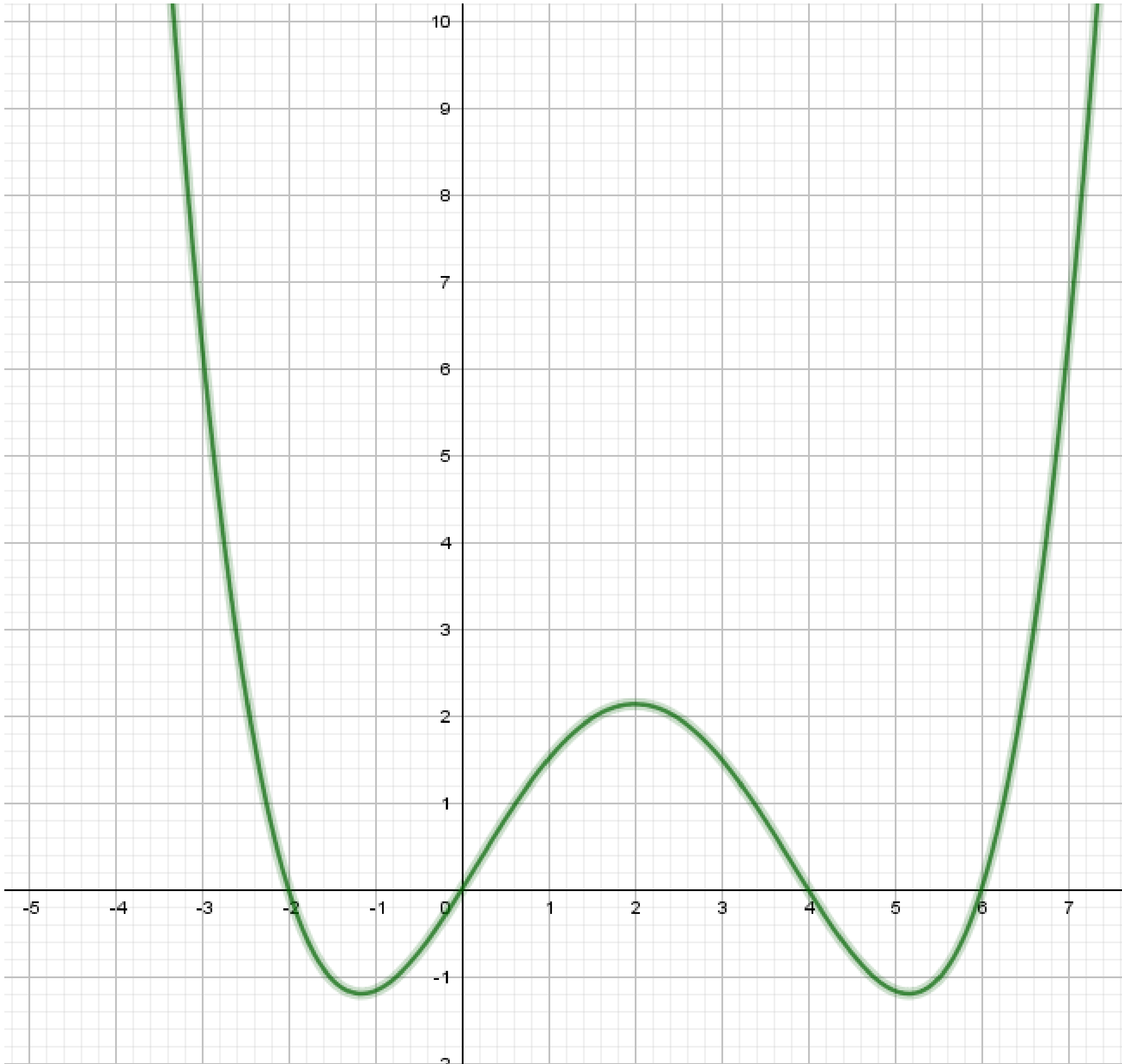
Example 1:

Given the graph below, determine the x-values determine:

A) The intervals where the function is concave up

B) The intervals where the function is concave down

C) All inflection points (if any)



Solution:

We should see that the first part of the graph starts as a valley until about (0.25, 0.5).

It then switches back from hill to valley at about (3.5, 0.5).

It then continues on a valley through to the end of the graph.

This means that the function is concave up on $(-\infty, 0.5), (3.5, \infty)$ and is concave down on $(0.5, 3.5)$.

Since the graph changes concavity at $x = 0.25$ and $x = 3.5$, the two inflection points are (0.25, 0.5) and (3.5, 0.5).

Examples: Concavity

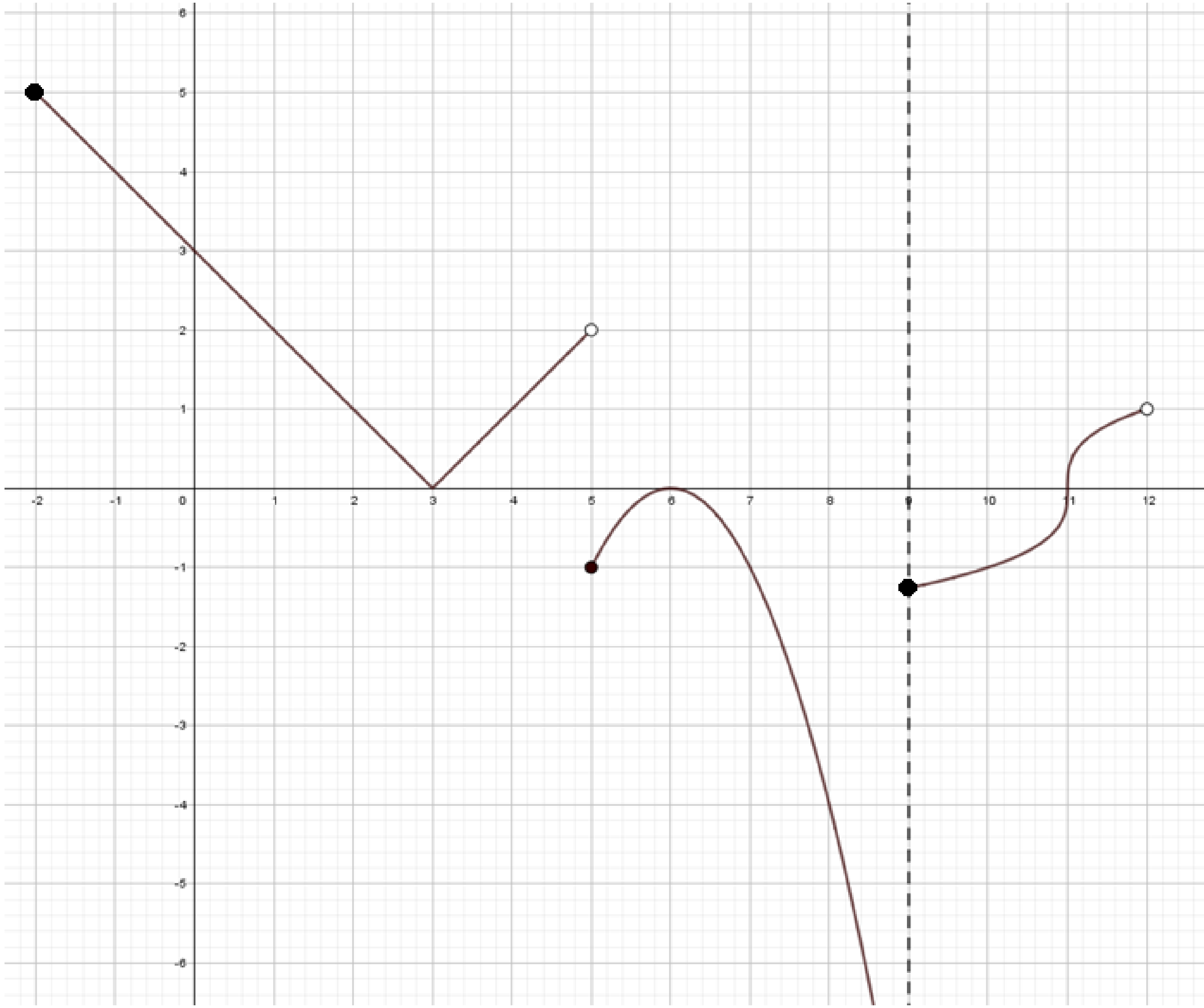
Example 2:

Given the graph below, determine the x-values determine:

A) The intervals where the function is concave up

B) The intervals where the function is concave down

C) All inflection points (if any)



Solution:

- The first part of the graph is linear, so there is no concavity on linear pieces.
- From 5 to 9 the graph is on a hill, so this is concave down.
- From 9 to 11 the graph is on a valley, so this is concave up.
- Finally from 11 to 12, the function is on a hill, so this part is concave down.

This means the function is concave down on $(5,9), (11,12)$ and the function is concave up on $(9,11)$.

Since the function changes concavity $x = 11$ and the function is continuous there, there is an inflection point at $(11,0)$. We note that at $x = 9$ the function does change concavity, but since it is not continuous there, we do not say this is an inflection point.

Strategy: Finding intervals of Concavity and Inflection Points

How To Use it:	When To Use it:	Why this works?
To find the intervals of increase/decrease we can: 1) Identify the domain of the function. 2) Identify all x values for the points of discontinuity. 3) Determine $f''(x)$. 4) Factor $f''(x)$ fully and find all possible inflection points x-values (where the second derivative is 0 or undefined). 5) Create an interval table for $f''(x)$ (not $f(x)$ nor $f'(x)$) where we plot the domain, points of discontinuity, and potential inflection points as the column separators, and the factors of the derivative as the rows. The product will tell us if the value is positive or negative which determines if the function is concave up (+) or concave down (-). 6) If a point switches concavity (and is not a point of discontinuity) then those are the inflection points. NOTE: We list the intervals rather than use the union symbol to avoid confusion about "continuing concavity".	When we want to find when a function is concave up or concave down and/or find inflection points.	To see a proof of this, we can visit here.

Examples: Concavity

Example 3:

Determine the intervals of concave up and concave down for a function f that has a domain of $(-\infty, \infty)$, is continuous everywhere, and a derivative given as: $f'(x) = \frac{3x^2}{(3x^2+1)^2}$. Also determine any x −values where the function has inflection points.

Solution:

We know that there are no points of discontinuity (given in the question), so we simply find the second derivative. We are already given f' so we just need to derive again:

$$
\begin{aligned}
f''(x) \quad &= \frac{6x\left(3x^2+1\right)^2 - 3x^2\left(2\left(3x^2+1\right)(6x)\right)}{(3x^2+1)^4} \\
&= \frac{\left(3x^2+1\right)\left[6x\left(3x^2+1\right) - 36x^3\right]}{(3x^2+1)^4} \\
&= \frac{6x - 18x^3}{(3x^2+1)^3} \\
&= \frac{6x[1-3x^2]}{(3x^2+1)^3}
\end{aligned}
$$

Here we see that we do not have any undefined derivatives (denominator is never zero), but we do see that the numerator will be 0 when $x = 0$ or when $x = \pm\sqrt{1/3}$

Examples: Concavity

Example 3 (cont):

Determine the intervals of concave up and concave down for a function f that has a domain of $(-\infty, \infty)$, is continuous everywhere, and a derivative given as: $f'(x) = \frac{3x^2}{(3x^2+1)^2}$. Also determine any x –values where the function has inflection points.

Solution:

Finally we can create an interval table to determine the areas of concavity and also find inflection points:

$-\infty \qquad -\sqrt{\frac{1}{3}} \qquad 0 \qquad \sqrt{\frac{1}{3}} \qquad \infty$

f' factors	-1	-0.001	0.001	1
$6x$	-	-	+	+
$1 - 3x^2$	-	+	+	-
$3x^2 + 1$	+	+	+	+
product	+	-	+	-

Thus the function is concave up on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$ and on $\left(0, \sqrt{\frac{1}{3}}\right)$, it is concave down on $\left(-\sqrt{\frac{1}{3}}, 0\right)$ and on $\left(\sqrt{\frac{1}{3}}, \infty\right)$ and it has 3 inflection points that occur at x-values of $-\sqrt{\frac{1}{3}}$, 0, and $\sqrt{\frac{1}{3}}$ as the concavity changes at these three points (and the function is continuous at these points as well).

Strategy: Classifying Critical Points using the Second Derivative

How To Use it:	When To Use it:	Why this works?
If we have found the critical point at $x = a$, and we have the second derivative of the function $f''(x)$. Then if we sub in $f''(a)$ we get: 1) $f''(a)$ is positive means the critical point is a local minimum. 2) $f''(a)$ is negative means the critical point is a local maximum. 3) $f''(a)$ is undefined or 0 means the test fails.	When we want to know if a critical point is a local maximum or minimum and we know the second derivative.	If the second derivative is positive at the critical point then the critical point must be on a valley and thus is a minimum point. Similarly, if the second derivative is negative it must be on a hill and thus a maximum point.

Example 5:
Explain what you can do to classify a critical point if the second derivative test fails?

Solution:
You can use the first derivative test instead.

Examples: Concavity

Example 6:

Consider a function that has four critical points: $(2,5)$, $(7,8)$, $(-2,-4)$, and $(-8,8)$. It is known that the second derivative is given by: $f''(x) = \frac{x-2}{\sqrt[3]{x+8}}$

Classify all points using the second derivative test (or state why it is not possible to do so).

Solution:

We sub in the x-values into the second derivative to determine if we can use the second derivative test:

$(2,5) \quad \rightarrow f''(2) = \frac{2-2}{\sqrt[3]{2+8}} = 0$ Thus the second derivative test fails and we do not know if this is a local max or min.

$(7,8) \quad \rightarrow f''(7) = \frac{7-2}{\sqrt[3]{7+8}} = +$ Thus the second derivative test tells us that this critical point is a local minimum.

$(-2,-4) \rightarrow f''(-2) = \frac{-2-2}{\sqrt[3]{-2+8}} = -$ Thus the second derivative test tells us that this critical point is a local maximum.

$(-8,8) \quad \rightarrow f''(-8) = \frac{-8-2}{\sqrt[3]{-8+8}} = und$ Thus the second derivative test fails and we do not know if this is a local max or min.

Strategy: Finding Limits using l'hospital's Rule

L'hopitals Rule:

Given a limit of the form $\lim_{x \to a} \frac{N(x)}{D(x)}$ such that it goes to $\frac{\infty}{\infty}$ or $\frac{0}{0}$. Then we have a new limit $\lim_{x \to a} \frac{N'(x)}{D'(x)}$ that also has the same limit as the original $\lim_{x \to a} \frac{N(x)}{D(x)}$.

How To Use it:

Check if the limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then derive the numerator and derive the denominator separately to get a new limit. Find the value of this limit to find the value of the original limit.

Note: It is possible that:

1) **You may need/want to use l'hopitals rule again on the new limit (which is allowed as long as the new limit is $\frac{0}{0}$ or $\frac{\infty}{\infty}$).**
2) **It is possible that the new limit is a lot more difficult to deal with than the first limit. In this case it is better to explore other limit strategies instead of l'hopital's rule.**

When To Use it:

When calculating a limit of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then we can try to use l'hopital's rule.

Why this works?

To see a proof of l'hopital's rule, click here.

Examples: l'Hopital's Rule

Example 1:

Determine the value of the following limit: $\lim_{x\to\infty} \frac{x^3-8x^2+x}{3-x^3}$

Solution:

We first note that this limit will be of a form $\frac{\infty}{\infty}$ thus we can use l'hospital's rule:

$$\lim_{x\to\infty} \frac{x^3-8x^2+x}{3-x^3} = \lim_{x\to\infty} \frac{3x^2-16x+1}{-3x^2}$$

This is still $\frac{\infty}{\infty}$ so we can use l'hopitals again.

$$= \lim_{x\to\infty} \frac{6x-16}{-6x}$$

This is still $\frac{\infty}{\infty}$ so we can use l'hopitals again.

$$= \lim_{x\to\infty} \frac{6}{-6}$$

$$= -1$$

Thus our limit is -1

Examples: l'Hopital's Rule

Example 2:

Determine the value of the following limit: $\lim_{x \to 1} \frac{\sqrt{5x-1}-2x}{x-1}$

Solution:

We first note that this limit will be of a form $\frac{0}{0}$ thus we can use l'hospital's rule:

$$
\begin{aligned}
\lim_{x \to 1} \frac{\sqrt{5x-1}-2x}{x-1} \qquad & = \lim_{x \to 1} \frac{\frac{1}{2\sqrt{5x-1}}(5)-2}{1} \\
& = \frac{1}{2\sqrt{5(1)-1}}(5) - 2 \\
& = \frac{5}{4} - 2 \\
& = -\frac{3}{4}
\end{aligned}
$$

Thus our limit is $-\frac{3}{4}$

Examples: l'Hopital's Rule

Example 3:

Determine the value of the following limit: $\lim_{x \to 0} \frac{x - \sin(x)}{x^3}$

Solution:

We first note that this limit will be of a form $\frac{0}{0}$ thus we can use l'hospital's rule:

$$\lim_{x \to 0} \frac{x - \sin(x)}{x^3} = \lim_{x \to 0} \frac{1 - cos(x)}{3x^2}$$

This is still $\frac{0}{0}$ so we can use l'hopitals again.

$$= \lim_{x \to 0} \frac{sin(x)}{6x}$$

This is still $\frac{0}{0}$ so we can use l'hopitals again.

$$= \lim_{x \to 0} \frac{cos(x)}{6}$$

$$= \frac{1}{6}$$

Thus our limit is $\frac{1}{6}$

Examples: l'Hopital's Rule

Example 4:

Determine the value of the following limit: $\lim_{x\to 2} \frac{2^x-3^x+5}{x-2}$

Solution:

We first note that this limit will be of a form $\frac{0}{0}$ thus we can use l'hospital's rule:

$$\lim_{x\to 2} \frac{2^x-3^x+5}{x-2} \qquad = \lim_{x\to 2} \frac{\ln(2)2^x-\ln(3)3^x}{1}$$
$$= 4\ln(2) - 9\ln(3)$$

Thus our limit is $4\ln(2) - 9\ln(3)$

Strategy: 0 Times ∞

How To Use it:	When To Use it:	Why this works?
If we have the indeterminant form $0 \times \infty$, then you can try: 1) If we have AB, then rewrite the expression as $\frac{A}{B^{-1}}$ or $\frac{B}{A^{-1}}$. 2) This will change the limit to $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so you can use l'hopital's rule.	When calculating a limit of the form $0 \times \infty$, then we can try to use l'hopital's rule.	We know from fractions that AB is equal to $\frac{A}{B^{-1}}$ and $\frac{B}{A^{-1}}$. But rewriting this in a new way allows us to get $\frac{\infty}{\infty}$ or $\frac{0}{0}$ so we can use l'hopital's rule.

Example 5:

Determine the value of the following limit: $\lim_{x \to \infty} xe^{-x}$

Solution:

We first note that this limit will be of a form $\infty \times 0$ thus we must rearrange the equation before using l'hopital's rule:

$$\begin{aligned}
\lim_{x\to\infty} xe^{-x} &= \lim_{x\to\infty} \frac{x}{e^{-(-x)}} \\
&= \lim_{x\to\infty} \frac{x}{e^{x}} \qquad \text{(This is now } \tfrac{\infty}{\infty}\text{, thus we can use l'hopital's rule)} \\
&= \lim_{x\to\infty} \frac{1}{e^{x}} \\
&= \frac{1}{\infty} \\
&= 0
\end{aligned}$$

Examples: 0 times ∞

Example 6:

Determine the value of the following limit: $\lim_{x \to 0^+} x \ln(x)$

Solution:

We first note that this limit will be of a form $0 \times \infty$ thus we must rearrange the equation before using l'hopital's rule:

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-1}}$$

(This is now $\frac{\infty}{\infty}$ so we can use l'hopital's rule)

$$= \lim_{x \to 0^+} \frac{1/x}{-x^{-2}}$$

$$= \lim_{x \to 0^+} -x$$

$$= 0$$

Examples: 0 times ∞

Example 7:

Determine the value of the following limit: $\lim_{x\to 0^+} x^3 e^{x^{-1}}$

Solution:

We first note that this limit will be of a form $0 \times \infty$ thus we must rearrange the equation before using l'hopital's rule:

$$\lim_{x\to 0^+} x^3 e^{x^{-1}} = \lim_{x\to 0^+} \frac{e^{x^{-1}}}{x^{-3}}$$

(This is now $\frac{\infty}{\infty}$ so we can use l'hopital's rule)

$$= \lim_{x\to 0^+} \frac{-x^{-2}e^{x^{-1}}}{-3x^{-4}}$$

$$= \lim_{x\to 0^+} \frac{e^{x^{-1}}}{3x^{-2}}$$

(This is still $\frac{\infty}{\infty}$ so we can use l'hopital's rule again)

$$= \lim_{x\to 0^+} \frac{-x^{-2}e^{x^{-1}}}{-6x^{-3}}$$

$$= \lim_{x\to 0^+} \frac{e^{x^{-1}}}{6x^{-1}}$$

(This is still $\frac{\infty}{\infty}$ so we can use l'hopital's rule again)

$$= \lim_{x\to 0^+} \frac{-x^{-2}e^{x^{-1}}}{-6x^{-2}}$$

$$= \lim_{x\to 0^+} \frac{e^{x^{-1}}}{6}$$

$$= \frac{\infty}{6}$$

$$= \infty$$

Examples: 0 times ∞

Example 8:

What happens if we do the example again but take the reciprocal of the other piece? $\lim_{x \to 0^+} x^3 e^{x^{-1}}$

Explain why this will not give us a reasonable path to the answer.

Solution:

$$\lim_{x \to 0^+} x^3 e^{x^{-1}} = \lim_{x \to 0^+} \frac{x^3}{e^{-x^{-1}}}$$ (This is now $\frac{0}{0}$ so we can use l'hopital's rule)

$$= \lim_{x \to 0^+} \frac{3x^2}{x^{-2} e^{-x^{-1}}}$$

$$= \lim_{x \to 0^+} \frac{3x^4}{e^{-x^{-1}}}$$ (This is still $\frac{0}{0}$ so we can use l'hopital's rule again)

We can see that we are permitted to keep going using l'hopital's rule, but the numerator expression got larger in degree. If we continue to do l'hopital's rule, the denominator will continue to get larger and degree and this process will never terminate. Thus we should not have manipulated the product in this way if we hope to get to an answer.

Strategy: $0^0, 1^\infty, \infty^0$

How To Use it:	When To Use it:	Why this works?
If we have an indeterminant form $\lim_{x\to a} f(x)$ of form 0^0, 1^∞ or ∞^0, we can: 1) Let $L = \lim_{x\to a} \ln(f(x))$ 2) The limit will now be of the form $0 \times \infty$. Manipulate the limit to use l'hopital's rule. 3) Once you find the answer for new limit L, your final answer for the original limit will be e^L.	When our limit is of the form 0^0, 1^∞ or ∞^0.	Log brings exponents down in front of the log allowing us to manipulate the expression to use l'hopital's rule. We then need to e the result to undo the log we imposed when we started the question.

Example 9:

Determine the value of the following limit: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Solution:

We first note that this limit will be of a form 1^∞ thus we use the ln and e method:

Let L

$$= \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x$$

$$= \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}}$$ (This is now $\frac{0}{0}$ so we can use l'hopital's rule)

$$= \lim_{x\to\infty} \frac{\frac{1}{1+x^{-1}}\left(-x^{-2}\right)}{-x^{-2}}$$

$$= \lim_{x\to\infty} \frac{1}{1+x^{-1}}$$

$$= \frac{1}{1+0}$$

$$= 1$$

Thus our limit is $e^L = e^1 = e$.

Examples: $0^0, 1^\infty, \infty^0$

Example 10:

Determine the value of the following limit: $\lim_{x \to 1^+} x^{1/(x-1)}$

Solution:

We first note that this limit will be of a form 1^∞ thus we use the ln and e method:

Let L

$$
\begin{aligned}
&= \lim_{x \to 1^+} \ln\left(x^{1/(x-1)}\right) \\
&= \lim_{x \to 1^+} \frac{1}{x-1} \ln(x) \\
&= \lim_{x \to 1^+} \frac{\ln(x)}{x-1} \qquad \text{(This is now } \tfrac{0}{0} \text{ so we can use l'hopital's rule)} \\
&= \lim_{x \to 1^+} \frac{1/x}{1} \\
&= \lim_{x \to 1^+} \frac{1}{x} \\
&= 1
\end{aligned}
$$

Thus our limit is $e^L = e^1 = e$.

Examples: $0^0, 1^\infty, \infty^0$

Example 11:

Determine the value of the following limit: $\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{1/x}$

Solution:

We first note that this limit will be of a form ∞^0 thus we use the ln and e method:

Let L $= \lim_{x\to\infty} \ln\left(\frac{x^2+1}{x+2}\right)^{1/x}$

$$= \lim_{x\to\infty} \frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)$$

$$= \lim_{x\to\infty} \frac{\ln(x^2+1)-\ln(x+2)}{x}$$ (This is now $\frac{\infty}{\infty}$ so we can use l'hopital's rule)

$$= \lim_{x\to\infty} \frac{\frac{1}{x^2+1}(2x)-\frac{1}{x+2}(1)}{1}$$

$$= \lim_{x\to\infty} \frac{2x}{x^2+1}-\frac{1}{x+2}$$

$$= \lim_{x\to\infty} \frac{2x(x+2)-(x^2+1)}{(x^2+1)(x+2)}$$

$$= \lim_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2}$$ (This is now $\frac{\infty}{\infty}$ so we can use l'hopital's rule)

$$= \lim_{x\to\infty} \frac{2x+4}{3x^2+4x+1}$$ (This is now $\frac{\infty}{\infty}$ so we can use l'hopital's rule)

$$= \lim_{x\to\infty} \frac{2}{6x+4}$$ (This is now $\frac{\infty}{\infty}$ so we can use l'hopital's rule)

$$= \frac{2}{\infty}$$
$$= 0$$

Thus our limit is $e^L = e^0 = 1.$

Definition: Cycling

There are cases where l'hopital's rule will fail because of one of the following:

1) The expression is not $\frac{\infty}{\infty}$ nor $\frac{0}{0}$ and cannot be manipulated to be as such.

2) The derivatives make the expression enormous and are impractical to deal with.

3) The derivatives will never get out of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (we call this idea **cycling**)

In these instances, we need to use our other limit strategies to solve the problem or try a manipulation (in cases 2 and 3) that reduces the derivative or changes the expression so l'hopital's rule can be used more effectively.

Example 12:

Identify the error in the following "solution".

Then create the correct solution afterwards.

$$\lim_{x\to 0}\frac{xe^x}{x^3+x^2+x} = \lim_{x\to 0}\frac{e^x+xe^x}{3x^2+2x+1}$$

$$= \lim_{x\to 0}\frac{e^x+e^x+xe^x}{6x+2}$$

$$= \lim_{x\to 0}\frac{e^x+e^x+e^x+xe^x}{6}$$

$$= \frac{1+1+1+0}{6}$$

$$= \frac{1}{2}$$

Solution:

In two separate instances we used l'hopital's rule when the limit was not of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form. Instead we should get:

$$\lim_{x\to 0}\frac{e^x+xe^x}{3x^2+2x+1} = \frac{1+0}{0+0+1} = 1$$

Examples: Cycling

Example 13:

Show that l'hopital's rule could be used in the following limit and then use it.

Explain the issue that arises and determine the limit through manipulating the limit before using l'hopital's rule: $\lim_{x\to 0^+} \frac{x}{e^{-1/x}}$

Solution:

We should see that the numerator goes to 0 and the denominator goes to $e^{-\infty} = 0$. Thus this is of the form $\frac{0}{0}$. However, when we use l'hopital's rule we get:

$$\lim_{x\to 0^+} \frac{x}{e^{-1/x}} = \lim_{x\to 0^+} \frac{1}{x^{-2}e^{-1/x}}$$

$$= \lim_{x\to 0^+} \frac{x^2}{e^{-1/x}}$$

The problem ended up getting into a larger problem than the one we started with. It is still of the form $\frac{0}{0}$, but even if we do l'hopital's rule again we get:

$$= \lim_{x\to 0^+} \frac{2x}{x^{-2}e^{-1/x}}$$

$$= \lim_{x\to 0^+} \frac{2x^3}{e^{-1/x}}$$

We should see that continuing on this path will not allow for us to arrive at an answer (we will always get to $\frac{0}{0}$ form). However, if we switch the numerator with the denominator in a "creative" way we get:

$$\lim_{x\to 0^+} \frac{x}{e^{-1/x}} = \lim_{x\to 0^+} \frac{e^{1/x}}{x^{-1}}$$

(This is now $\frac{\infty}{\infty}$ so we can still use l'hopital's rule)

$$= \lim_{x\to 0^+} \frac{-x^{-2}e^{1/x}}{-x^{-2}}$$

$$= \lim_{x\to 0^+} e^{1/x}$$

$$= \infty$$

Examples: Cycling

Example 14:

Show that l'hopital's rule could be used in the following limit and then use it.

Explain the issue that arises and determine the limit through manipulating the limit before using l'hopital's rule: $\lim_{x\to\infty}\frac{2^x+3^x}{4^x+5^x}$

Solution:

Here we see that we have $\frac{\infty}{\infty}$ which means we can use l'hopital's rule.

$$\lim_{x\to\infty}\frac{2^x+3^x}{4^x+5^x} \qquad = \lim_{x\to\infty}\frac{\ln(2)2^x+\ln(3)3^x}{\ln(4)4^x+\ln(5)5^x}$$

The problem ended up getting into a larger problem than the one we started with. It is still of the form $\frac{\infty}{\infty}$, but even if we do l'hopital's rule again we get:

$$= \lim_{x\to\infty}\frac{[[\ln(2)]^2 2^x+[[\ln(3)]^2 3^x}{[[\ln(4)]^2 4^x+[[\ln(5)]^2 5^x}$$

We should see that continuing on this path will not allow for us to arrive at an answer (we will always get to $\frac{\infty}{\infty}$ form).

However, if we divide everything by the highest term in the denominator we would get:

$$\lim_{x\to\infty}\frac{2^x+3^x}{4^x+5^x} \qquad = \lim_{x\to\infty}\frac{\frac{2^x}{5^x}+\frac{3^x}{5^x}}{\frac{4^x}{5^x}+\frac{5^x}{5^x}}$$

$$= \lim_{x\to\infty}\frac{\left(\frac{2}{5}\right)^x+\left(\frac{3}{5}\right)^x}{\left(\frac{4}{5}\right)^x+1}$$

Since we know that $(a)^x \to 0$ when $|a| < 1$ and when $x \to \infty$, we get:

$$= \frac{0+0}{0+1} \qquad = 0$$

Strategy: Finding Vertical Asymptotes and Holes

How To Use it:

To find vertical asymptotes/holes we:

1) Find all points of discontinuity (division by zero, logs, tan, change in piecewise, endpoints, etc…). Call any such point "a"
2) Test the limit at "a". Note that it may be needed to test the right hand and left hand limits for each point of discontinuity:

$$\lim_{x \to a^+} f(x) = L_1 \text{ and } \lim_{x \to a^-} f(x) = L_2$$

3) If one of the limits (or both) go to a number L_1 (or L_2), this produces a hole at the point (a, L_1) (or a hole at (a, L_2) or possibly both).

4) If one of the limits (or both) go to infinity (or negative infinity), then this produces a vertical asymptote (on one or both sides of a depending on the limits).

When To Use it:

When determining the location and nature of vertical asymptotes and/or holes.

Why this works?

This is simply applying the definition of what it means to be a vertical asymptote.

Examples: Vertical Asymptotes and Holes

Example 1:

Determine the location of any hole(s) or vertical asymptote(s) that appear in the function below. If the function has a vertical asymptote, describe what is happening on either side of the asymptote. $y = \frac{x^2-1}{x^2-3x+2}$

Solution:

We first note that the denominator would be zero if $x^2 - 3x + 2 = (x-2)(x-1) = 0$. This means we have two possible areas of holes/asymptotes: $x = 1$ or $x = 2$. We test each x value separately:

$\underline{x = 1}$

If we sub in $x = 1$ we get $\frac{0}{0}$ so we can use l'hopital's rule:

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{2x}{2x-3} = \frac{2(1)}{2(1)-3} = -2$$

Thus we have a hole at $x = 1$ as the limit went to a number -2.

The hole would be at $(1, -2)$

$\underline{x = 2}$

If we sub in $x = 2$ we get $\frac{3}{0}$ so we must check left and right limits to determine the behaviour of the infinity:

$$\lim_{x\to 2^+} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 2^+} \frac{(x-1)(x+1)}{(x-1)(x-2)} = \lim_{x\to 2^+} \frac{(x+1)}{(x-2)} = \frac{3^+}{0^+} = \infty$$

$$\lim_{x\to 2^-} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 2^-} \frac{(x-1)(x+1)}{(x-1)(x-2)} = \lim_{x\to 2^-} \frac{(x+1)}{(x-2)} = \frac{3^-}{0^-} = -\infty$$

Since at least one side (in this case both sides) approach an infinity, we have an asymptote at x=2. On the left side of the asymptote, the function approaches $-\infty$ and on the right side the function approaches ∞.

Examples: Vertical Asymptotes and Holes

Example 2:

Determine the location of any hole(s) or vertical asymptote(s) that appear in the function below. If the function has a vertical asymptote, describe what is happening on either side of the asymptote. $f(x) = x\ \ln|x|$

Solution:

We first note that there is only one point of discontinuity at $x = 0$ as we cannot log 0 (we also cannot have negatives, but the absolute value fixes that for us in this instance). We next note that this limit will be of a form $\infty \times 0$ thus we must rearrange the equation before using l'hopital's rule:

$$\lim_{x\to 0} x\ \ln|x| = \lim_{x\to 0} \frac{\ln|x|}{x^{-1}} \qquad \text{(This is now } \tfrac{\infty}{\infty} \text{ so we can use l'hopital's rule)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{|x|}\left(\frac{x}{|x|}\right)}{-x^{-2}}$$

$$= \lim_{x\to 0} \frac{\frac{x}{|x|^2}}{-x^{-2}}$$

$$= \lim_{x\to 0} \frac{\frac{x}{x^2}}{-x^{-2}} \qquad \text{(As } |x|^2 = x^2 \text{)}$$

$$= \lim_{x\to 0} \frac{x^{-1}}{-x^{-2}}$$

$$= \lim_{x\to 0} -x$$

$$= 0$$

Thus our limit is 0 and so we have a hole at $x = 0$. This means the hole happens at $(0,0)$.

Note that there are no vertical asymptotes as we only had one point of discontinuity.

Examples: Vertical Asymptotes and Holes

Example 3:

Determine the location of any hole(s) or vertical asymptote(s) that appear in the function below. If the function has a vertical asymptote, describe what is happening on either side of the asymptote. $f(x) = xe^{x^{-1}}$

Solution:

We have one point of discontinuity at $x = 0$ due to the x^{-1} that appears. We also note that when $x \to 0^+$ vs $x \to 0^-$ we have $e^{x^{-1}}$ behaves quite differently, so we will need to consider a limit on each side:

$\underline{x \to 0^+}$

We first note that this limit will be of a form $\infty \times 0$ thus we must rearrange the equation before using l'hopital's rule:

$$\lim_{x\to 0^+} xe^{x^{-1}} = \lim_{x\to 0^+} \frac{e^{x^{-1}}}{x^{-1}} \quad \text{(This is now } \tfrac{\infty}{\infty} \text{ so we can use l'hopital's rule)}$$

$$= \lim_{x\to 0^+} \frac{-x^{-2}e^{x^{-1}}}{-x^{-2}}$$

$$= \lim_{x\to 0^+} e^{x^{-1}}$$

$$= \infty$$

Thus we have an asymptote at $x = 0$ approaching on the right hand side.

$\underline{x \to 0^-}$

$$\lim_{x\to 0^-} xe^{x^{-1}} = (0^-)(e^{-\infty})$$

$$= (0)(0)$$

$$= 0$$

Thus we have also have a hole at $x = 0$ approaching on the left hand side. The hole will appear at (0,0)

Strategy: Finding Horizontal Asymptotes

How To Use it:	When To Use it:	Why this works?
To find horizonal asymptotes we: 1) Find the limits as $x \to \infty$ and as $x \to -\infty$. 2) If the limit goes to a number L, then we have a horizontal asymptote on that side (it is possible to have two different horizontal asymptotes one for each side). If it goes to ∞ or $-\infty$ then there is no horizontal asymptote.	When determining if a function has horizontal asymptotes.	This is simply applying the definition of what it means to be a horizontal asymptote.

Example 4:

Determine the any horizontal asymptote(s) that appear in the function below. $y = \frac{x^2-1}{x^2-3x+2}$

Solution:

We take a limit as $x \to \infty$ and $x \to -\infty$ separately to see if there are asymptotes on either side of the function.

$\underline{x \to \infty}$

$$\lim_{x\to\infty} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to\infty} \frac{2x}{2x-3} \quad \text{(since it is } \tfrac{\infty}{\infty} \text{ we can use l'hopital's rule)}$$

$$= \lim_{x\to\infty} \frac{2}{2} \quad \text{(since it is } \tfrac{\infty}{\infty} \text{ we can use l'hopital's rule)}$$

$$= 1$$

$\underline{x \to -\infty}$

$$\lim_{x\to-\infty} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to-\infty} \frac{2x}{2x-3} \quad \text{(since it is } \tfrac{\infty}{\infty} \text{ we can use l'hopital's rule)}$$

$$= \lim_{x\to-\infty} \frac{2}{2} \quad \text{(since it is } \tfrac{\infty}{\infty} \text{ we can use l'hopital's rule)}$$

$$= 1$$

This means we have one horizontal asymptote, and the function approaches $y = 1$ both as it tends to infinity and as it tends to negative infinity.

Examples: Horizontal Asymptotes

Example 5:

Determine the any horizontal asymptote(s) that appear in the function: $y = \frac{5}{1+e^{-2x}}$

Solution:

We take a limit as $x \to \infty$ and $x \to -\infty$ separately to see if there are asymptotes on either side of the function.

$\underline{x \to \infty}$

$$\lim_{x\to\infty} \frac{5}{1+e^{-2x}} = \frac{5}{1+0} \quad \text{(since } e^{-\infty} \to 0\text{)}$$
$$= 5$$

$\underline{x \to -\infty}$

$$\lim_{x\to-\infty} \frac{5}{1+e^{-2x}} = \frac{5}{1+\infty} \quad \text{(since it is } e^{\infty} \to \infty\text{)}$$
$$= 0$$

This means we have two horizontal asymptotes, and the function approaches $y = 5$ as x tends to ∞ and approaches $y = 0$ as x tends to $-\infty$.

Examples: Horizontal Asymptotes

Example 6:

Determine the any horizontal asymptote(s) that appear in the function: $f(x) = xe^{x^{-1}}$

Solution:

We take a limit as $x \to \infty$ and $x \to -\infty$ separately to see if there are asymptotes on either side of the function.

$\underline{x \to \infty}$

$$\lim_{x\to\infty} xe^{x^{-1}} = \infty\left(e^{\frac{1}{\infty}}\right)$$

$$= \infty(e^0) \qquad \text{(since } \tfrac{1}{\infty} \to 0\text{)}$$

$$= \infty$$

$\underline{x \to -\infty}$

$$\lim_{x\to-\infty} xe^{x^{-1}} = -\infty\left(e^{\frac{1}{-\infty}}\right)$$

$$= -\infty(e^0) \qquad \text{(since } \tfrac{1}{\infty} \to 0\text{)}$$

$$= -\infty$$

Thus this function has no horizontal asymptotes.

Strategy: Curve Sketching

How To Use it:	When To Use it:	Why this works?
To sketch the curve we should: 1) Plot all of the key points given: endpoints, intercepts, local max/min, holes, inflection points, vertical and horizontal asymptotes. 2) For each vertical asymptote, determine the behaviour on either side using increasing and decreasing and plot a "short arrow" do indicate what is happening on each side of vertical asymptote. 3) For each horizontal asymptote, determine whether the function is coming in on top or below the asymptote by looking at increasing and decreasing. 4) For each critical point/inflection point/asymptote arrow to the next, determine the concavity increasing/decreasing nature in that interval, and connect the pieces together. When connecting these key points, try to go through the correct value of the x and/or y intercept as well.	When asked to determine the sketch of a curve and you are given the following information: 1) Domain 2) Intercepts and endpoints 3) Holes 4) Local max and mins 5) Inflection points 6) Vertical Asymptotes 7) Horizontal Asymptotes 8) Intervals of Increasing and Decreasing 9) Intervals of Concavity	This is simply providing a set algorithm to take all that was learned about functions to sketch the curve.

Curve Sketching Helpful Hints

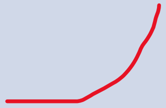
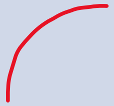
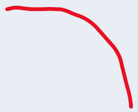
Vertical Asymptotes:

- Increasing towards a vertical asymptote means going to infinity on the left side, and decreasing towards an asymptote means going to infinity on the left side.
- Increasing starting from a vertical asymptote means negative infinity on the right side of the asymptote, and decreasing starting from an asymptote means going to infinity on the right side of the asymptote.

Horizontal Asymptotes:

- Increasing from $-\infty$ when there is a horizontal asymptote on the left side of the graph means the function is going above the asymptote on the left side. Decreasing from $-\infty$ when there is an asymptote on the left side of the graph means the function is going below the asymptote on the left side.
- Increasing towards ∞ when there is an asymptote on the right side of the graph means the function is going below the asymptote on the right side. Decreasing towards ∞ when there is an asymptote on the right side of the graph means the function is going above the asymptote on the right side.

Connecting pieces together using increasing/decreasing and concavity:

	Concave Up	Concave Down
Increasing		
Decreasing		

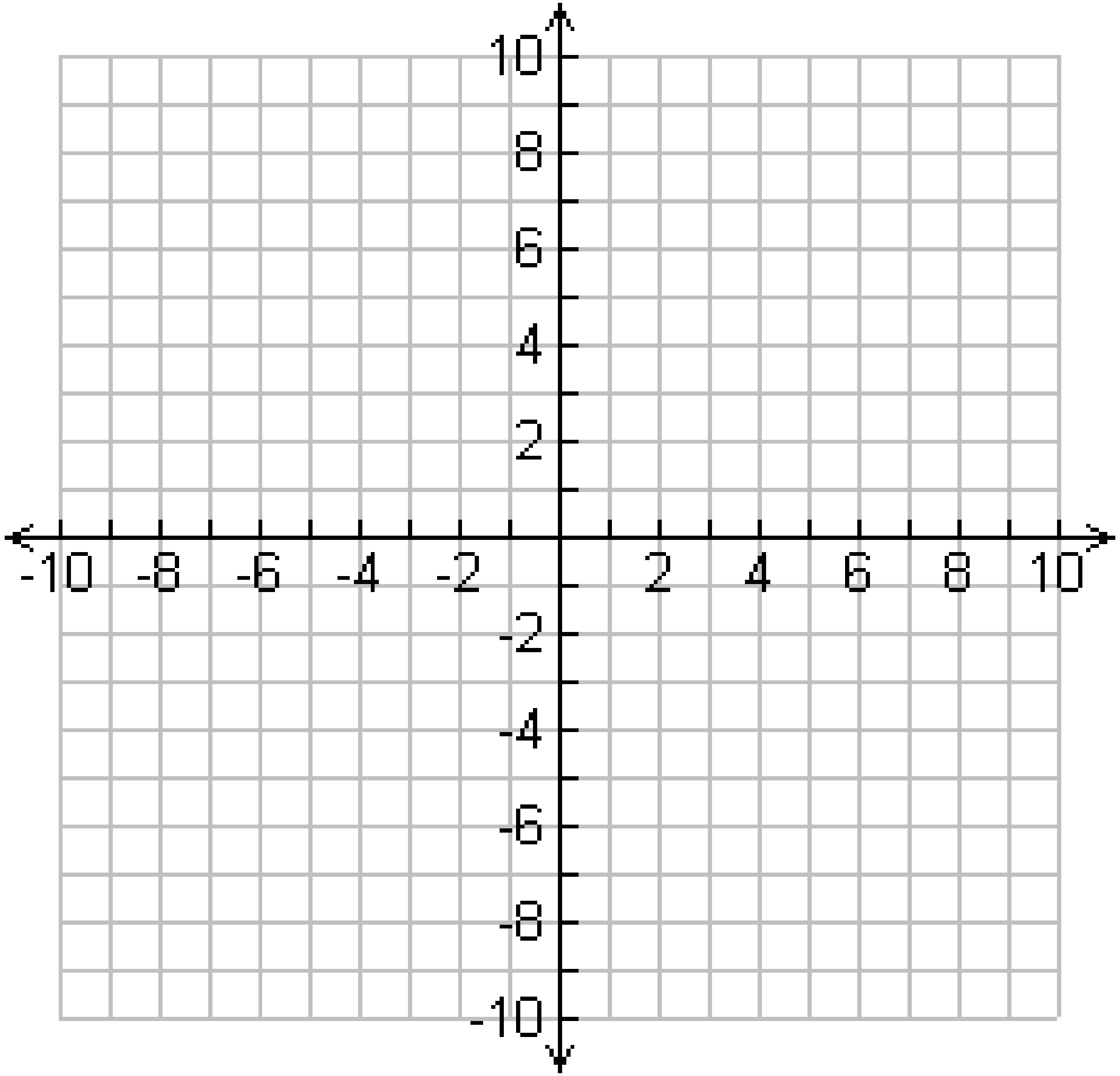
Examples: Curve Sketching

Example 7:

Sketch the curve using the following information:

Consider the function $f(x) = (x^2 - 4)^{2/3}$

Domain	$(-\infty, \infty)$
Intercepts and endpoints	Y –int 2.5 X – ints -2 and 2
Holes	None
Local max and mins	Local Max (0, 2.5) Local Mins (-2,0) and (2,0)
Inflection points	(-3.5,4) and (3.5,4)
Vertical Asymptotes	None
Horizontal Asymptotes	None
Intervals of Increasing and Decreasing	Decreasing on $(-\infty - 2)$ and on $(0,2)$, Increasing on $(-2,0)$ and on $(2, \infty)$
Intervals of Concavity	Concave up on $(-\infty, -3.5)$ and on $(3.5, \infty)$ Concave down on $(-3.5, -2), (-2,2)$ and on $(2,3.5)$



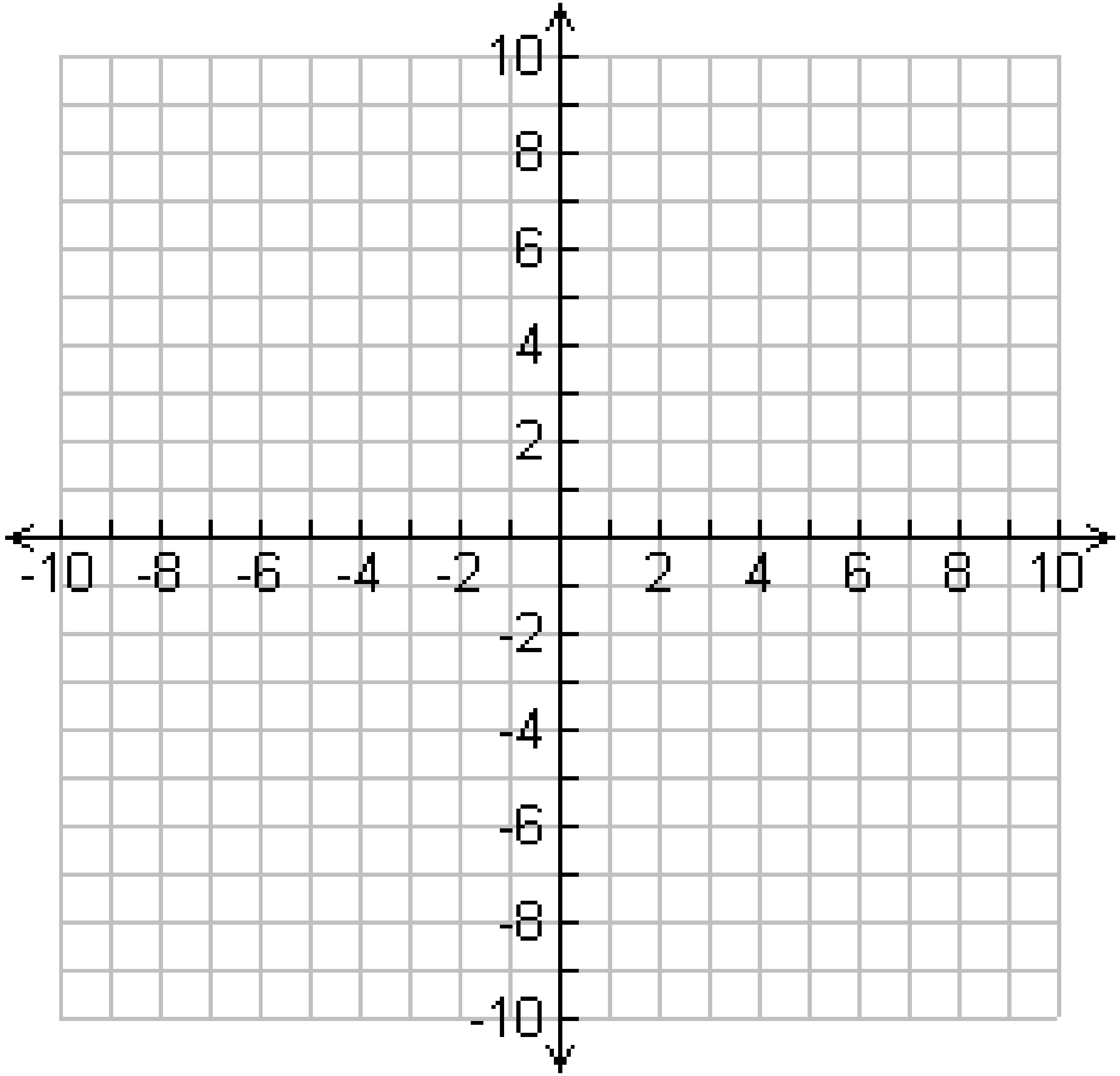
Examples: Curve Sketching

Example 8:

Sketch the curve using the following information:

Consider the function $f(x) = 20xe^{-x}$

Domain	$(-\infty, \infty)$
Intercepts and endpoints	Y – int 0 X – int 0
Holes	None
Local max and mins	Local Max (1, 7.4)
Inflection points	(2,5.4)
Vertical Asymptotes	None
Horizontal Asymptotes	As $x \to \infty$ we have HA at $y = 0$
Intervals of Increasing and Decreasing	Increasing on $(-\infty 1)$ Decreasing on $(1, \infty)$
Intervals of Concavity	Concave up on $(2, \infty)$ Concave down on $(-\infty, 2)$



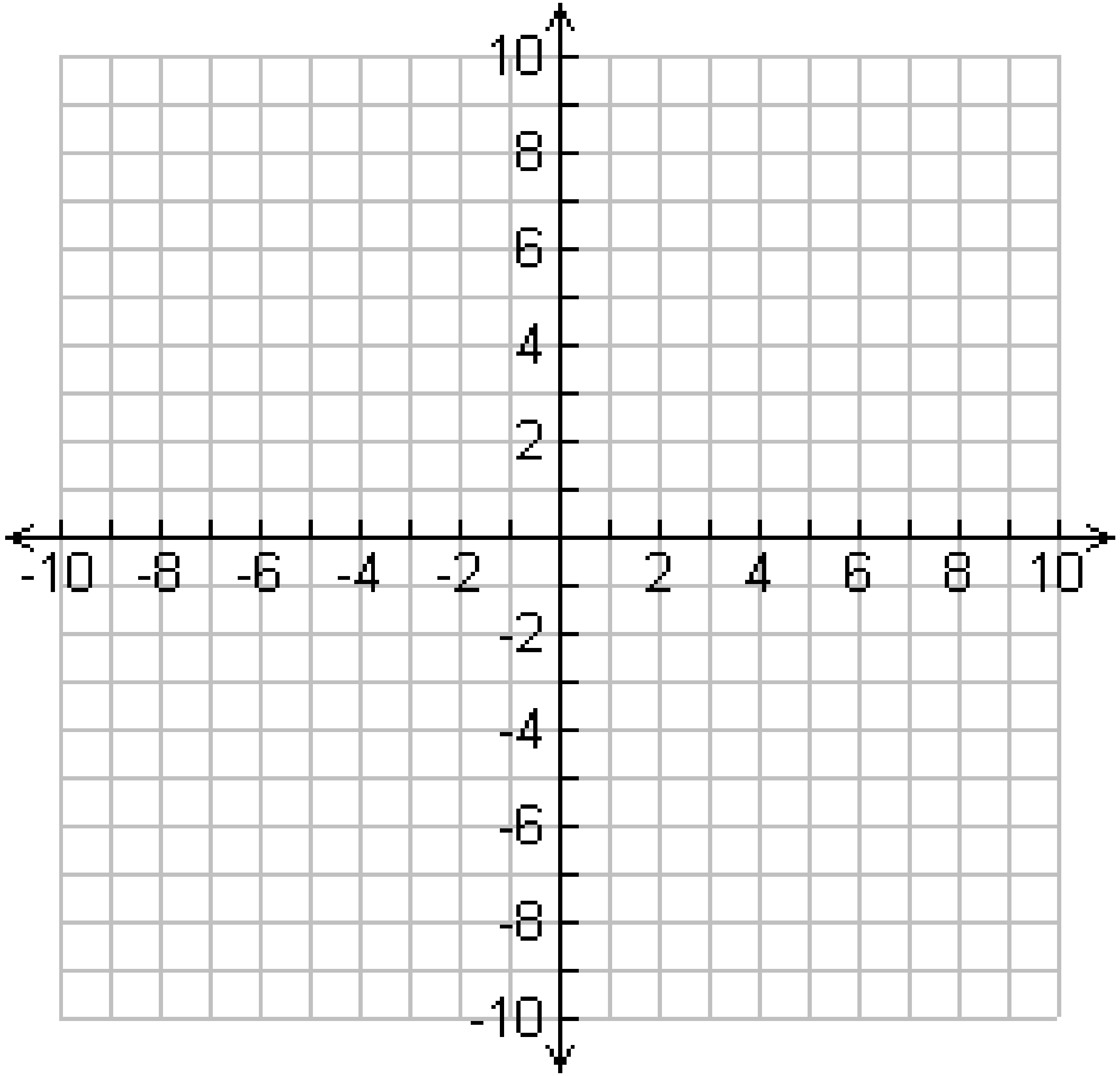
Examples: Curve Sketching

Example 9:

Sketch the curve using the following information:

Consider the function $f(x) = \frac{6}{1+e^{-x}}$

Domain	$(-\infty, \infty)$
Intercepts and endpoints	X – int: none Y – int: 3
Holes	none
Local max and mins	none
Inflection points	(0,3)
Vertical Asymptotes	none
Horizontal Asymptotes	As x→ ∞ we have a HA at $y = 6$ As $x \to -\infty$ we have a HA at $y = 0$
Intervals of Increasing and Decreasing	Increasing $(-\infty, \infty)$ Decreasing: never
Intervals of Concavity	Concave up on $(-\infty, 0)$ Concave down on $(0, \infty)$



Examples: Curve Sketching

Example 10:

Sketch the curve using the following information on the interval $[-10,10]$

Consider the function $f(x) = 50\frac{1-\cos(x)}{x^2-\frac{\pi}{2}x}$

Domain	$[-10,10]$
Intercepts and endpoints	X – int: , $(-6.5,0), (6.5,0)$, Y – int: none Endpoints: (-10,0.5) (10,1)
Holes	(0,0)
Local max and mins	Local Max (-9,1), (-1.5, 10), (9,1.5) Local Min (-6.5,0), (6.5,0)
Inflection points	(-8,0.5), (-3,7), (7.5,0.5)
Vertical Asymptotes	x= 1.5
Horizontal Asymptotes	Cannot consider as we are in a domain of $[-10,10]$ (if we considered the whole domain, we would have $y = 0$ as a horizontal asymptote).
Intervals of Increasing and Decreasing	Increasing on $(-10,-9), (-6.5,-1.5)$ and on $(6.5,9)$ Decreasing on $(-9,-6.5), (-1.5,1.5), (1.5,6.5)$ and on $(9,10)$
Intervals of Concavity	Concave up on $(-8,-3) \cup (1.5,7.5)$ Concave down on $(-10,-8) \cup (-3,1.5) \cup (7.5,10)$